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Erratum: Stable and convergent fully discrete interior–exterior coupling of Maxwell's equations

Jörg Nick · Balázs Kovács · Christian Lubich

Abstract We correct a sign error in the paper [3] by the second and third authors, noted by the first author. This sign error in the definition of the Calderón operator has no effect on the theory presented in [3], but it does affect the implementation of the proposed numerical method.

Keywords transparent boundary conditions · boundary integral equations · Calderón operator

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1 Introduction

In [3] we present a time-domain boundary integral formulation of an interior–exterior coupling of Maxwell's equations, with the help of a Calderón operator whose coercivity plays a fundamental role in proving the well-posedness of the proposed time-domain boundary integral equations and the stability of the numerical discretization. The definition of the Calderón operator contains, however, a sign error, which is corrected here. The effects of this sign error are restricted only to Section 2.3 and formula (3.1) in [3], but otherwise all the results of the paper hold unchanged. On the other hand, for the implementation of the method the correct sign is crucial.

J. Nick, B. Kovács, and Ch. Lubich
Mathematisches Institut, University of Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: {nick,kovacs,lubich}@na.uni-tuebingen.de

2 The time-harmonic Maxwell's equation and its boundary integral operators

2.1 Time-harmonic Maxwell's equation and trace operators

Let us recall the *time-harmonic Maxwell's equation*, obtained as the Laplace transform of the second order Maxwell's equation (with constant permeability μ and permittivity ε):

$$\varepsilon\mu s^2 u + \operatorname{curl}\operatorname{curl} u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \Gamma, \tag{2.1}$$

where Γ is the boundary of a bounded piecewise smooth domain (or a finite collection of such domains) $\Omega \subset \mathbb{R}^3$, not necessarily convex, with exterior normal ν. The complex parameter s of positive real part is the Laplace transform variable.

In the following we assume appropriate physical units such that

$$\varepsilon\mu = 1, \tag{2.2}$$

that is, the wave speed is set to one. In the original work [3], the dependence on $\varepsilon\mu$ is unreliable and one should assume (2.2), which just corresponds to a rescaling of time $t \to t/\sqrt{\varepsilon\mu}$ or of frequency $s \to s\sqrt{\varepsilon\mu}$.

With the scaling (2.2), Eq. (2.1) becomes the time-harmonic Maxwell's equation $\operatorname{curl}\operatorname{curl} u - \kappa^2 u = 0$ as in [2] on setting $s = -i\kappa$.

Analogously to [2] the *tangential* and *magnetic* traces are defined by

$$\gamma_T v = v|_\Gamma \times \nu, \qquad \text{and} \qquad \gamma_N v = (s^{-1} \operatorname{curl} v)|_\Gamma \times \nu,$$

respectively. The setting uses the following skew-hermitian pairing on $L^2(\Gamma)$:

$$[\gamma\, w, \gamma\, v]_\Gamma = \int_\Gamma (\gamma\, \overline{w} \times \nu) \cdot \gamma\, v \,\mathrm{d}\sigma.$$

The complex conjugation of w was missing in the definition of the pairing in [3] although it was actually used, e.g. in formula (2.3) and Lemma 3.1 of [3].

2.2 Boundary integral operators

The functional analytic setting of [3, Section 2.3] follows Buffa and Hiptmair [2]. The latter paper defines boundary integral operators in the Fourier domain, whereas [3] uses the Laplace domain (which fits better with convolution quadratures, cf. [3, Section 4]). The sign error occurred while translating the definition of the boundary integral operators and related notions from the Fourier to the Laplace domain. Below we present the correct Laplace domain formulation.

Exactly as in [3], following [2] and [1], the (electric) *single layer potential* and *double layer potential* for (2.1) are given, for $x \in \mathbb{R}^3 \setminus \Gamma$, as

$$\begin{aligned}
\mathcal{S}(s)\varphi(x) = & -s \int_\Gamma G(s, x-y)\varphi(y)\mathrm{d}y + s^{-1}\nabla \int_\Gamma G(s, x-y)\,\mathrm{div}_\Gamma\,\varphi(y)\mathrm{d}y, \\
\mathcal{D}(s)\psi(x) = & \ \mathrm{curl} \int_\Gamma G(s, x-y)\psi(y)\mathrm{d}y,
\end{aligned}$$

with the fundamental solution $G(s,x) = \frac{e^{-s|x|}}{4\pi|x|}$ for $x \in \mathbb{R}^3 \setminus \{0\}$ and $\mathrm{Re}\, s > 0$.

The solution of (2.1) is then given by the correct representation formula:

$$u = -\mathcal{S}(s)\varphi + \mathcal{D}(s)\psi, \qquad x \in \mathbb{R}^3 \setminus \Gamma. \tag{2.3}$$

In [3] the first negative sign was erroneously missing.
The boundary densities in (2.3) are given by $\varphi = [\![\gamma_N u]\!] = [\![\gamma_T(s^{-1}\,\mathrm{curl}\, u)]\!]$ and $\psi = [\![\gamma_T u]\!]$, where $[\![\gamma v]\!] = \gamma^- v - \gamma^+ v$ denotes the jump in the boundary traces of the interior domain Ω^- and the exterior domain Ω^+, while $\{\!\{\gamma v\}\!\} = \frac{1}{2}(\gamma^- v + \gamma^+ v)$ denotes the average. We note that there is a sign difference in the jump when comparing [2] and [3].

Due to the negative sign in the representation formula the correct jump relations are

$$\begin{aligned}
[\![\gamma_N \circ \mathcal{S}(s)]\!] &= -\,\mathrm{Id}, & \qquad [\![\gamma_N \circ \mathcal{D}(s)]\!] &= 0, \\
[\![\gamma_T \circ \mathcal{S}(s)]\!] &= 0, & \qquad [\![\gamma_T \circ \mathcal{D}(s)]\!] &= \mathrm{Id}.
\end{aligned}$$

The boundary integral operators V and K then satisfy the relations

$$\begin{aligned}
V(s) &= \{\!\{\gamma_T \circ \mathcal{S}(s)\}\!\} = \{\!\{\gamma_N \circ \mathcal{D}(s)\}\!\}, \\
K(s) &= \{\!\{\gamma_T \circ \mathcal{D}(s)\}\!\} = -\{\!\{\gamma_N \circ \mathcal{S}(s)\}\!\}.
\end{aligned} \tag{2.4}$$

In [3] the negative sign in the last term of the second line was missing. Naturally, this sign difference does not influence the boundedness of these operators, see [3, Lemma 2.3], based on [2, Section 5] and [1].

The negative sign in (2.4) changes the signs in the expression for the averages of the traces using the operators V and K, see [3, equation (2.6)]. The correct relations are:

$$\begin{aligned}
\{\!\{\gamma_T u\}\!\} &= -\,\{\!\{\gamma_T \mathcal{S}(s)\varphi\}\!\} + \{\!\{\gamma_T \mathcal{D}(s)\psi\}\!\} \\
&= -\,V(s)\varphi + K(s)\psi, \qquad\qquad \text{and} \\
\{\!\{\gamma_N u\}\!\} &= -\,\{\!\{\gamma_N \mathcal{S}(s)\varphi\}\!\} + \{\!\{\gamma_N \mathcal{D}(s)\psi\}\!\} \\
&= K(s)\varphi + V(s)\psi.
\end{aligned} \tag{2.5}$$

The negative sign in the first equation was missing in [3, equation (2.6)].

3 Coercivity of a Calderón operator for the time-harmonic Maxwell's equation

Due to the above formulas, the correct *Calderón operator* is given by

$$B(s) = \mu^{-1} \begin{pmatrix} -V(s) & K(s) \\ -K(s) & -V(s) \end{pmatrix}, \tag{3.1}$$

with a correct negative sign in the left upper block of $B(s)$ as opposed to [3, equation (3.1)].

Within the above setting the first equality in the proof of Lemma 3.1 in [3] stays true: For given $\varphi, \psi \in \mathcal{H}_\Gamma$, we define $u \in H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)$ by the representation formula (2.3). We can then express φ and ψ, see above, by $\varphi = [\![\gamma_N u]\!] = [\![\gamma_T(s^{-1}\,\mathrm{curl}\, u)]\!]$ and $\psi = [\![\gamma_T u]\!]$. Then, (2.5) and (3.1) yield

$$B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mu^{-1} \begin{pmatrix} \{\!\{\gamma_T u\}\!\} \\ -\{\!\{\gamma_N u\}\!\} \end{pmatrix}. \tag{3.2}$$

Remark 3.1 When the scaling (2.2) is not imposed, then the corresponding equation is obtained by replacing the argument s with $s\sqrt{\varepsilon\mu}$ in $B(s)$ and in $\gamma_N u = \gamma_T(s^{-1}\,\mathrm{curl}\, u)$. We note, however, that with this substitution, the single- and double-layer operators are then scaled differently from those defined in [1,3].

It is of crucial importance that in the above setting the Calderón operator still satisfies the following coercivity result, with the proof given as in [3].

Lemma 3.1 ([3, Lemma 3.1]) *There exists* $\beta > 0$ *such that the Calderón operator* (3.1) *satisfies*

$$\mathrm{Re} \left[\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma \geq \beta\, m(s) \Big(\|s^{-1}\varphi\|^2_{\mathcal{H}_\Gamma} + \|s^{-1}\psi\|^2_{\mathcal{H}_\Gamma} \Big)$$

for $\mathrm{Re}\, s > 0$ *and for all* $\varphi, \psi \in \mathcal{H}_\Gamma$*, with* $m(s) = \min\{1, |s|^2\}\, \mathrm{Re}\, s$.

Thanks to this coercivity estimate for the Calderón operator B defined above in (3.1), all the stability and convergence results of [3] remain valid, since the proofs depend on this coercivity result and not on the particular form of the Calderón operator.

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